An overview of Domain Decomposition Methods for Maxwell

Victorita Dolean

Outline of my talk

1- Introduction and motivations

2 – Time harmonic Maxwell

3 – Optimised Schwarz methods

4 – Two-level preconditioners: the grid coarse space

5 - Conclusions and future work

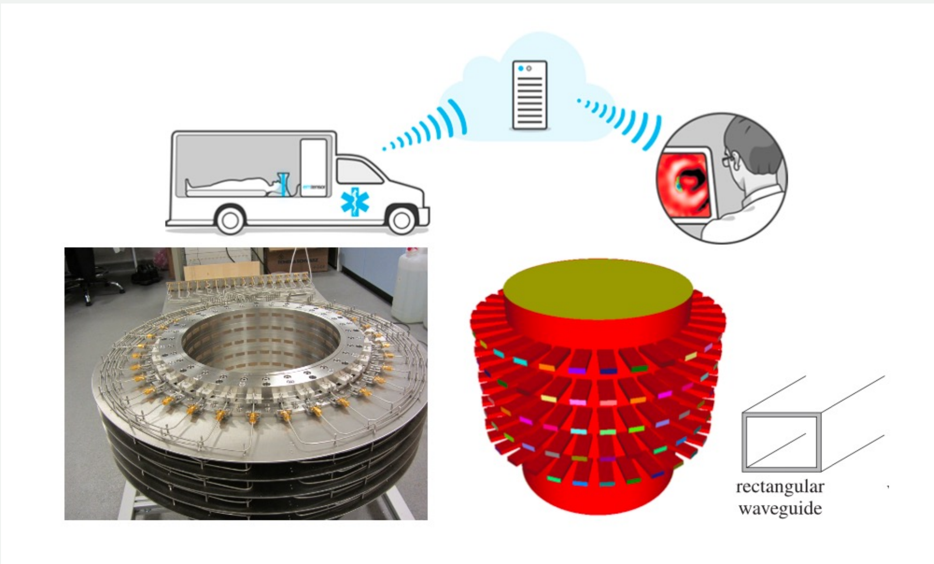
1 - Introduction: Maxwell's equations in heterogeneous media

AIM: propose a combination of discretisation + solvers in time and frequency domain.

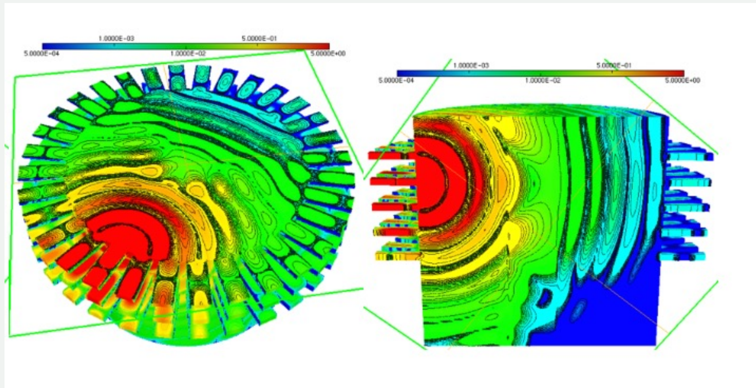
- Space discretisation : use volumic discretisations, e.g. H-curl conforming edge-elements, preferably higher order.
- **Time-harmonic Maxwell's equations**: natural formulation for problems in science and engineering. Downsides: they lead to non-selfadjoint problems, very difficult for solvers in high frequency regime
- **Time discretised Maxwell's equations**: SPD. Downsides: difficulties persist due to the large kernel of the curl operator: difficult to design 2-level methods.

1 –Motivation: an example of application (time-harmonic)

Microwave imaging system prototype (EMTensor):
Cylindrical chamber with 5 rings of 32 *rectangular waveguides*



Each waveguide alternately transmits an electromagnetic wave, 160 receiving waveguides: repeated solutions of a Maxwell problem with different rhs.



1 – Introduction and motivation

James Clerk Maxwell

Maxwell's equations in time domain

$$-\varepsilon\frac{\partial\mathcal{E}}{\partial t}+\nabla\times\mathcal{H}-\sigma\mathcal{E}=\mathcal{J},\qquad \mu\frac{\partial\mathcal{H}}{\partial t}+\nabla\times\mathcal{E}=0,$$

Time-harmonic Maxwell's equations ($\mathcal{J}(\boldsymbol{x},t)=\mathcal{R}e(\boldsymbol{J}(\boldsymbol{x})\exp(i\omega t))$)

$$-i\omega\varepsilon\mathbf{E}+\nabla\times\mathbf{H}-\sigma\mathbf{E}=\mathbf{J},\qquad i\omega\mu\mathbf{H}+\nabla\times\mathbf{E}=\mathbf{0}.$$

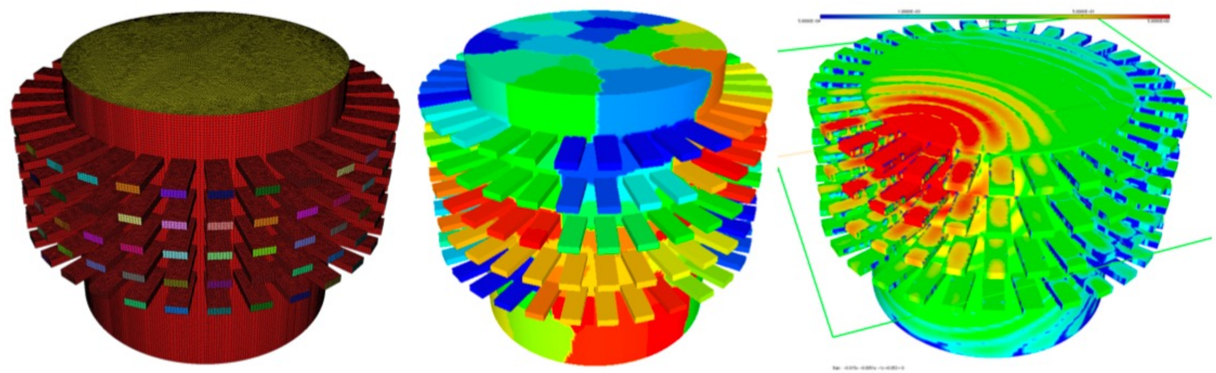
Eliminating $\boldsymbol{H}$ in this system of equations, we obtain the second order Maxwell equations If we suppose that $\sigma=0$ and μ constant then

$$\boxed{-\tilde{\omega}^2\boldsymbol{E}+\nabla\times(\nabla\times\boldsymbol{E})=-i\tilde{\omega}Z\boldsymbol{J},\ \tilde{\omega}=\omega\sqrt{\varepsilon\mu},\ Z=\sqrt{\frac{\mu}{\varepsilon}}.}$$

2 - Time harmonic problems

- Size increasing with the frequency (rule of thumb: 10 ppwl or more if er want to avoid pollution)
- **Ideas**: (1) change the interface conditions, (2) design two-level methods.

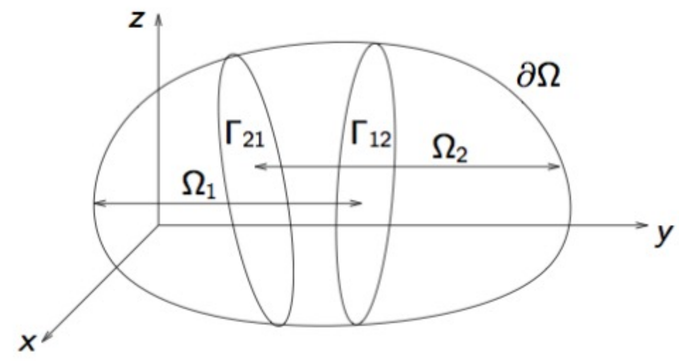
Spatial discretization using Nédélec edge finite elements yields a large sparse linear system $Au = f_j$ for each transmitting antenna j.
We need a robust and efficient solver $\Longrightarrow$ use domain decomposition.



3 – Optimised Schwarz methods

Compute iteratively for $n = 1, 2 \ldots$ the solution

$$
\begin{aligned}
-\omega^2 \mathbf{E}^{1,n} + \nabla \times \left(\nabla \times \mathbf{E}^{1,n}\right) &= -i\omega Z \mathbf{J}, \Omega_1 \\
\mathcal{T}_{\mathbf{n}_1}\left(\mathbf{E}^{1,n}\right) &= \mathcal{T}_{\mathbf{n}_1}\left(\mathbf{E}^{2,n-1}\right), \Gamma_{12} \\
-\omega^2 \mathbf{E}^{2,n} + \nabla \times \left(\nabla \times \mathbf{E}^{2,n}\right) &= -i\omega Z \mathbf{J}, \Omega_2 \\
\mathcal{T}_{\mathbf{n}_2}\left(\mathbf{E}^{2,n}\right) &= \mathcal{T}_{\mathbf{n}_2}\left(\mathbf{E}^{1,n-1}\right), \Gamma_{21}
\end{aligned}
$$



We propose to modify only the transmission operators: $\mathcal{T}_{\mathbf{n}_j}, j = 1, 2$ are tangential possibly pseudo-differential operators.

$$
\boxed{
\begin{aligned}
\mathcal{T}_{\mathbf{n}}(\mathbf{E}) \quad = \quad & (I + \nu_1(\delta_1 \mathcal{S}_{TM} + \delta_2 \mathcal{S}_{TE}))(\mathbf{n} \times \nabla \times \mathbf{E}) \\
+ \quad & i\omega(-I + \nu_2(\delta_3 \mathcal{S}_{TM} + \delta_4 \mathcal{S}_{TE}))(\mathbf{n} \times (\mathbf{E} \times \mathbf{n})),
\end{aligned}
}
$$

where $\mathcal{S}_{TM} = \nabla_\tau \nabla_\tau \cdot$, $\mathcal{S}_{TE} = \nabla_\tau \times \nabla_\tau \times$ and τ is the tangential direction.

3 – Features of the method

- Tangential operators S_{TE} S_{TM} act only on the TE and TM components of the solution
- All previously defined operators fall into this category for particular values of parameters.
- New conditions can be re-written as perturbation of impedance conditions.

$$\boxed{\begin{aligned}\tilde{\mathcal{T}}_{\mathbf{n}}(\mathbf{E}) &= \mathbf{n} \times (\nabla \times \mathbf{E}) - i\omega(\mathbf{n} \times (\mathbf{E} \times \mathbf{n})) \\ &+ \frac{1}{\omega^2 - i\omega s^{tm}} \mathcal{S}_{TM}(\mathbf{n} \times (\nabla \times \mathbf{E})) + \frac{i}{\omega - is^{te}} \mathcal{S}_{TE}(\mathbf{n} \times (\mathbf{E} \times \mathbf{n})).\end{aligned}}$$

- Parameters can be optimised.

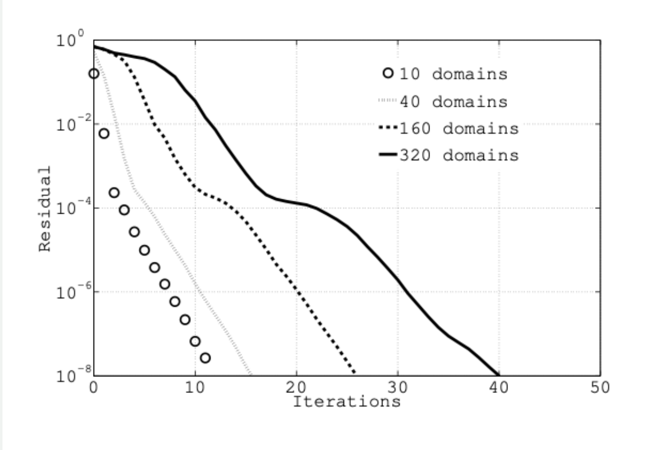
Effective transmission conditions for domain decomposition methods applied to the time-harmonic curl-curl Maxwell's equations

V Dolean, MJ Gander, S Lanteri, JF Lee, Z. Peng, JCP 2015

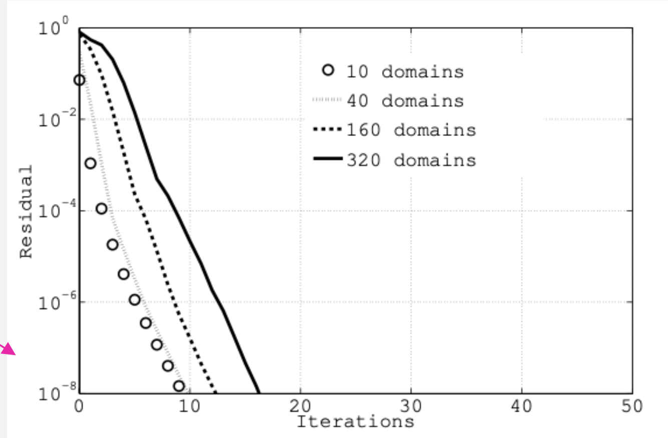
Scalability (2 variants of optimised conditions)

Scalability with respect to ωD:

- A WR-75 waveguide of length $0.0375m$ with mesh size $h = \lambda_0/4$
- Partitioned into 6 equally sized sub-domains of dimension $0.01905m \times 0.009595m \times 0.0125m$.
- The electric size of the waveguide increases accordingly four times.



Fixed subdomain size (1/4 pf wavelength): increase the number of subdomains by increasing the subdomain size.



Cases	$f = 12$ GHz	$f = 20$ GHz	$f = 30$ GHz	$f = 40$ GHz
RPL conditions	9 (13)	13 (23)	17 (39)	27 (51)
OSM conditions	12 (18)	14 (21)	13 (21)	15 (23)

4 – Two-level preconditioners: in a nutshell

Finding a robust solver for wave propagation problems is challenging

Sign-indefinite time-harmonic Maxwell problem

$$\begin{cases} \nabla \times (\nabla \times \mathbf{E}) - \tilde{\omega}^2 \mathbf{E} = \mathbf{F} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

Discretisation matrix A

The dissipative case should be easier...

Problem with absorption: matrix A_ξ

$$\begin{cases} \nabla \times (\nabla \times \mathbf{E}) - (\tilde{\omega}^2 + \mathrm{i}\xi)\mathbf{E} = \mathbf{F} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

Discretisation matrix A_ξ

Solve with GMRES

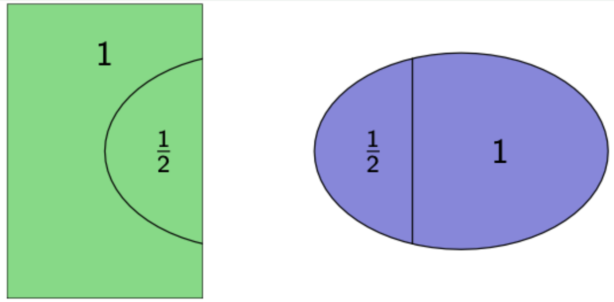
$$B_\xi^{-1} A\mathbf{x} = B_\xi^{-1}\mathbf{b},$$

where B_ξ^{-1} is an approximation of A_ξ^{-1} computed using DD

One level vs. two-level preconditioners

Consider the linear system: $Au = f \in \mathbb{C}^n$.

Given a decomposition of $[\![1;n]\!]$, $(\mathcal{N}_1, \mathcal{N}_2)$, define:
the restriction operator R_i from $[\![1;n]\!]$ into $\mathcal{N}_i$,
R_i^T as the extension by 0 from $\mathcal{N}_i$ into $[\![1;n]\!]$.



$A_i = R_i A R_i^T$

$$M_{\mathrm{AS}}^{-1} := \sum_{i=1}^{N} R_i^T A_i^{-1} R_i$$

$$M_{\mathrm{RAS}}^{-1} := \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i$$

$$M_{\mathrm{ORAS}}^{-1} := \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i$$

Add absorption and a second level (coarse space)

$$M_{\xi,\mathrm{AS}}^{-1} = \sum_{i=0}^{N_{\mathrm{sub}}} R_i^T (A_\xi^i)^{-1} R_i, \qquad A_\xi^i = R_i A_\xi R_i^T, \quad R_0 = Z^T$$

Here definition of Z based on a *coarser mesh* of element size H_{cs}:
the *interpolation matrix* from $V_{H_{\mathrm{cs}}}$ to V_h

Theoretical result – GMRES convergence, precond. system

Particular case: when $|\xi| \sim \tilde{\omega}^2$ (max absorption) and $\delta \sim H_{cs}$ (generous overlap),
Condition (ii) is satisfied with $H_{sub} \sim H_{cs} \sim \tilde{\omega}^{-1}$,
and then bound (iii) implies convergence *independent of* $\tilde{\omega}$.

Theorem (GMRES convergence for left preconditioning)

Ω convex polyhedron. Given $\tilde{\omega}_0 > 0$, there exists $\mathcal{C} > 0$, independent of all parameters, such that, given $0 < a < 1$, if

(i) $\tilde{\omega} \geq \tilde{\omega}_0$,

(ii) $\max\left\{(\tilde{\omega} H_{sub}),\ (\tilde{\omega} H_{cs})\left(\left(\frac{\tilde{\omega}^2}{|\xi|}\right)\right)\right\} \leq \mathcal{C}_1\left(1+\left(\frac{H_{cs}}{\delta}\right)^2\right)^{-1}\left(\frac{|\xi|}{\tilde{\omega}^2}\right)$,

(iii) $m \geq \mathcal{C}\left(\left(\frac{\tilde{\omega}^2}{|\xi|}\right)\right)^3\left(1+\left(\frac{H_{cs}}{\delta}\right)^2\right)\log\left(\frac{12}{a}\right)$,

then

$$\frac{\|\mathbf{r}_m\|_{D_{\tilde{\omega}}}}{\|\mathbf{r}_0\|_{D_{\tilde{\omega}}}} \leq a$$

Domain decomposition preconditioning for the high-frequency time-harmonic Maxwell equations with absorption
M Bonazzoli, V Dolean, I Graham, E Spence, PH Tournier, Mathematics of Computation, 2019.

Numerical results – not covered by the theory (I)

$H_{\text{sub}} \sim \tilde{\omega}^{-0.6}$

$H_{\text{cs}} \sim \tilde{\omega}^{-0.9}$

overlap $\delta \sim \cancel{H_{\text{sub}}}\ 2h$

$\xi_{\text{prob}} = \cancel{\tilde{\omega}^2}\ 0$, $\xi_{\text{prec}} = \cancel{\tilde{\omega}^2}\ \tilde{\omega}$

~~PEC~~ impedance B.Cs on $\partial\Omega$

order 1 edge FEs

$h \sim \tilde{\omega}^{-3/2}$

regular partitioning

			$\xi_{\text{prec}} = \tilde{\omega}$				
$\tilde{\omega}$	n	N_{sub}	2-level	n_{cs}	Time	1-level	Time
10	2.6×10^5	27	20	2.9×10^3	16.2(1.6)	37	13.7(2.6)
15	1.5×10^6	125	26	1.0×10^4	25.5(4.0)	70	26.1(8.9)
20	5.2×10^6	216	29	2.1×10^4	52.0(9.1)	94	60.6(25.6)
25	1.4×10^7	216	33	4.4×10^4	145.5(29.5)	105	191.2(88.1)
30	3.3×10^7	343	38	6.9×10^4	380.4(128.4)	132	673.5(443.1)

2-level: # OHRAS, 1-level: # ORAS, Total Time (GMRES Time)

Numerical results – not covered by the theory (II)

Order 2 FEs + fixed number of points per wavelength

$H_{\text{sub}} \sim \cancel{\tilde{\omega}^{-0.6}}\ (20\,\tilde{\omega}/(2\pi))^{-0.5}$

$H_{\text{cs}} \sim \cancel{\tilde{\omega}^{-0.9}}\ 2\pi/(2\,\tilde{\omega})$

overlap $\delta \sim 2h$

$\xi_{\text{prob}} = 0$, $\xi_{\text{prec}} = \tilde{\omega}$

impedance B.Cs on $\partial\Omega$

order $\cancel{1}$ 2 edge FEs

$h \sim \cancel{\tilde{\omega}^{-3/2}}\ 2\pi/(20\,\tilde{\omega})$

~~regular~~ METIS partitioning

			$g = 20,\ \alpha = 0.5,\ g_{cs} = 2$				
$\tilde{\omega}$	n	N_{sub}	2-level	n_{cs}	Time	1-level	Time
10	1.1×10^6	125	38	1.3×10^3	37.7(7.5)	80	36.6(14.8)
20	8.3×10^6	343	36	9.3×10^3	85.8(18.9)	123	161.7(72.1)
30	2.8×10^7	729	41	3.0×10^4	155.7(39.8)	162	267.3(174.5)
40	6.6×10^7	1331	51	7.0×10^4	272.3(77.6)	> 200	453.8(305.2)

2-level: # OHRAS, 1-level: # ORAS, Total Time (GMRES Time)

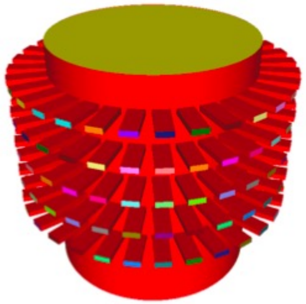
Medimax example

order 1 edge FEs, 40 points per wavelength

$n \approx 1.6 \times 10^7$

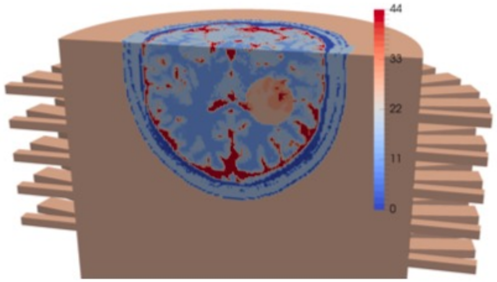
$n_{cs} \approx 3.8 \times 10^4$

729 subdomains



*

homogeneous gel



*

brain model

*

plastic cylinder

	#2-level	Time	#1-level	Time
homogeneous gel	28	63.4(8.6)	30	53.1(6.4)
brain model	28	64.1(9.2)	32	53.4(6.9)
non-conducting cylinder	29	62.3(9.4)	125	83.5(38.2)

COBRA cavity

order 2 edge FEs, 10 points per wavelength

coarse mesh: 3.33 points per wavelength

$\xi_{prob} = 0, \xi_{prec} = \tilde{\omega}$

$f = 10\ GHz$: $n \approx 1.07 \times 10^8$, $n_{cs} \approx 4 \times 10^6$

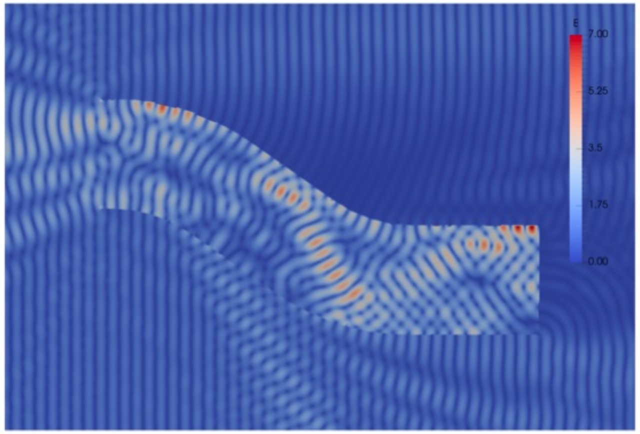
$f = 16\ GHz$: $n \approx 1.98 \times 10^8$, $n_{cs} \approx 7.4 \times 10^6$

$\rightarrow$ coarse problem too large for a direct solver

$\Rightarrow$ inexact coarse solve: use the one-level ORAS preconditioner

$\Rightarrow$ same decomposition for fine and coarse problems

16 GHz



		$g = 10, \quad g_{cs} = 3.33$					
			Total #	Total times (seconds)			
f	N_{sub}	# it	inner it	Total	Setup	GMRES	inner
10GHz	1536	134	4116	1128.9	415.7	713.1	309.3
10GHz	3072	139	6162	659.6	239.4	420.2	209.9
16GHz	3072	183	10645	1504.2	373.9	1130.3	568.5
16GHz	6144	198	14438	1037.0	267.7	769.4	433.8

speedup of 1.7 for $f = 10\ GHz$, 1.45 for $f = 16\ GHz$

Ongoing and future work

- Two-level methods for time-discretised Maxwell : ongoing
- Theory for impedance transmission conditions.
- PML in domain decomposition for Maxwell: ongoing
- Preprint(s) available shortly…

Thanks for your attention!