

# **Stratégie avec convergence garantie en décomposition de domaines pour les ondes avec présence de points de croisement.**

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# Scattering in heterogeneous medium

wave number :  $\kappa : \mathbb{R}^d \rightarrow \mathbb{C}$  bounded

$\Re\{\kappa(\mathbf{x})\} \geq 0, \Im\{\kappa(\mathbf{x})\} \geq 0, \kappa(\mathbf{x}) \neq 0$

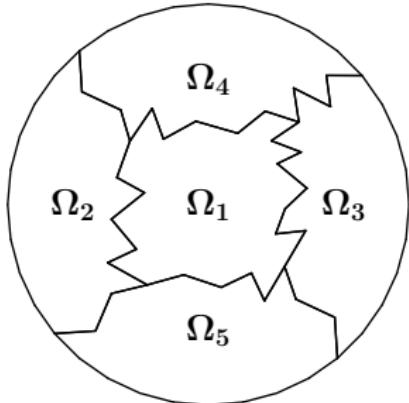
source :  $f \in L^2(\Omega)$

## Non-overlapping partition

$$\Omega = \overline{\Omega}_1 \cup \dots \cup \overline{\Omega}_J,$$

$$\Gamma_j := \partial\Omega_j, \quad \Gamma'_j := \Gamma_j \setminus \partial\Omega$$

$\Omega_j$  : Lipschitz, bounded



## Helmholtz bvp

$$-\Delta u - \kappa(\mathbf{x})^2 u = f \text{ in } \Omega,$$

$$\partial_n u - \imath\kappa u = 0 \text{ on } \partial\Omega.$$

$\iff$

## local sub-problems $j = 1 \dots J$

$$-\Delta u - \kappa^2 u = f \text{ in } \Omega_j$$

$$\partial_n u - \imath\kappa u = 0 \text{ on } \partial\Omega_j \cap \partial\Omega.$$

+



Cross points allowed

## transmission conditions

$$\partial_{n_j} u|_{\Gamma_j}^{\text{int}} = -\partial_{n_k} u|_{\Gamma_k}^{\text{int}}$$

$$u|_{\Gamma_j}^{\text{int}} = u|_{\Gamma_k}^{\text{int}} \quad \forall j, k$$

## Optimized Schwarz Method (OSM) [Després, 1991]

Optimized Schwarz Method (OSM) is one of the most established DDM approaches for wave propagation. This is a **substructuring method** where transmission conditions are imposed through each interface by means of **Robin traces** involving impedance coefficients.

- operator valued impedance : [Collino, Ghanemi & Joly, 2000]
- second order TC : [Gander, Magoules & Nataf, 2002]
- DtN-like impedance : [Nataf, Rogier & de Sturler, 1995], [Antoine, Boudendir & Geuzaine, 2012], [Antoine, Bouajaj & Geuzaine, 2014]
- large literature : overview article [Gander & Zhang, 2019]

### Cross point issue

Unappropriate treatment of cross-points may spoil convergence so care must be paid to this issue : [Gander & Kwok, 2013], [Gander & Santugini, 2016], [Després, Nicolopoulos & Thierry, 2020], [Modave, Antoine, Geuzaine & al, 2019 & 2020]. There is also a variant of FETI-DP "à la Després" [Farhat & al, 2005], [Bendali & Boubendir, 2006].

## **Outline**

**I Review of the Optimized Schwarz Method**

**II New manner to enforce transmission condition**

**III Numerical results**

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## Original OSM (Després)

**Transmission conditions** : with scalar  $\Lambda > 0$ ,

$$\begin{array}{lll} \partial_{\eta_j} u|_{\Gamma_j} = -\partial_{\eta_k} u|_{\Gamma_k} & \iff & +\partial_{\eta_j} u|_{\Gamma_j} + i\Lambda u|_{\Gamma_j} = \\ u|_{\Gamma_j} = u|_{\Gamma_k} & \iff & -\partial_{\eta_k} u|_{\Gamma_k} + i\Lambda u|_{\Gamma_k} \\ \text{on } \Gamma_j \cap \Gamma_k \forall j, k & & \text{on } \Gamma_j \cap \Gamma_k \forall j, k \end{array} \quad \boxed{\begin{array}{l} (\partial_{\eta_j} u|_{\Gamma'_j} - i\Lambda u|_{\Gamma'_j})_{j=1}^J = \\ -\Pi_0((\partial_{\eta_k} u|_{\Gamma'_k} + i\Lambda u|_{\Gamma'_k})_{k=1}^J) \end{array}}$$

where the operator  $\Pi_0$  swaps traces on both sides of each interfaces :

$$(v_0, \dots, v_J) = \Pi_0(u_0, \dots, u_J) \iff v_j = u_k \text{ on } \Gamma_j \cap \Gamma_k.$$

**Local scattering operators :**

$$S_0^{\Gamma_j}(\partial_{\eta_j} \psi|_{\Gamma'_j} - i\Lambda \psi|_{\Gamma'_j}) := \partial_{\eta_j} \psi|_{\Gamma'_j} + i\Lambda \psi|_{\Gamma'_j}$$

for  $\Delta \psi + \kappa^2 \psi = 0$  in  $\Omega_j$

$$\partial_n \psi - i\kappa \psi = 0 \text{ on } \partial \Omega_j \cap \partial \Omega$$

**Wave equations :**

$$\begin{array}{l} (\partial_{\eta_j} u|_{\Gamma'_j} + i\Lambda u|_{\Gamma'_j})_{j=1}^J = \\ S_0((\partial_{\eta_k} u|_{\Gamma'_k} - i\Lambda u|_{\Gamma'_k})_{k=1}^J) + g \end{array}$$

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stems from the source term of the bvp

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## Optimized Schwarz

$$(\text{Id} + \Pi_0 S_0)p = -\Pi_0(g)$$

$$\text{with } p = (\partial_{\eta_j} u|_{\Gamma'_j} - i\Lambda u|_{\Gamma'_j})_{j=1}^J$$

## The cross-point issue

$$p^{(n+1)} = (1 - r)p^{(n)} - r\Pi_0 S_0(p^{(n)}) - r\Pi_0(g)$$

Without cross points, geometric convergence can be obtained for appropriate (operator valued) impedance  $\Lambda$ . In practice, convergence is much slower with cross points i.e. at best algebraic  $\|p - p^{(n)}\|_{L^2} = O(n^{-\gamma})$ . The root cause seems related to  $\Pi_0$  not being continuous at cross-points in proper trace norms.

This so-called "cross point issue" also arises in the different context of multi-domain boundary integral formulations where Multi-Trace Formalism (MTF) [Claeys & Hiptmair, 2012] now offers a framework that accommodates cross-points, and that is clean as regards function spaces.

**Idea :** use the Multi-Trace Formalism to treat cross-points in OSM. We shall replace  $\Pi_0$  by a non-local counterpart  $\Pi$  that remains continuous no matter the presence of cross points, following the idea first introduced in :



X.Claeys, "Quasi-local multi-trace boundary integral formulations", Numer. Methods Partial Differential Equations, 31(6) :2043–2062, 2015.

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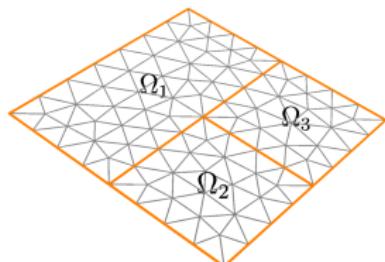
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## Discrete function spaces

Triangulation conforming with  $\Omega_j$ 's, FE spaces  $V_h(\Omega_j) = \{ \mathbb{P}_k\text{-Lagrange on } \Omega_j \}$ .

Volume functions	Tuples of traces ( $\Gamma_j := \partial\Omega_j$ )
$V_h(\Omega) := V_h(\Omega_1) \times \cdots \times V_h(\Omega_J)$	$\mathbb{V}_h(\Sigma) := V_h(\Gamma_1) \times \cdots \times V_h(\Gamma_J)$
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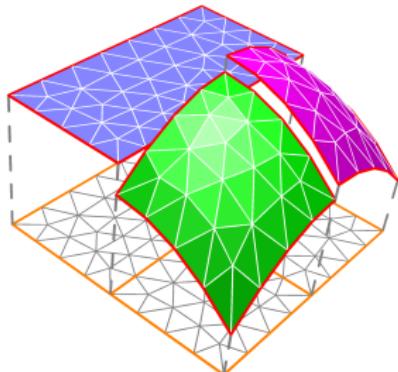


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Piecewise  $H^1$   
Possible jumps through interfaces

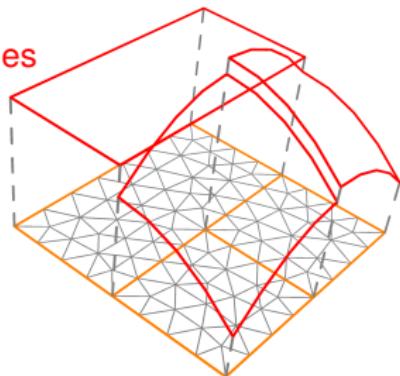


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**Multi-traces** = tuples of traces at boundaries

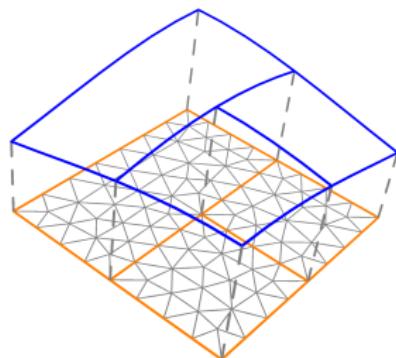


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**Single-traces** = tuples of traces  
that match at interfaces



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### Impedance

$$t_h(p, q) = t_{\Gamma_1}(p_1, q_1) + \cdots + t_{\Gamma_J}(p_J, q_J)$$

$t_{\Gamma_j}(\cdot, \cdot) = \text{any coercive sesquilinear form on } V_h(\Gamma_j)$

### Choices of impedance :

- surface mass matrix
- surface order 2 operator
- layer potential, DtN map
- Schur complements
- etc...

For the sake of clarity,  $t_h(\cdot, \cdot)$  is assumed SPD.

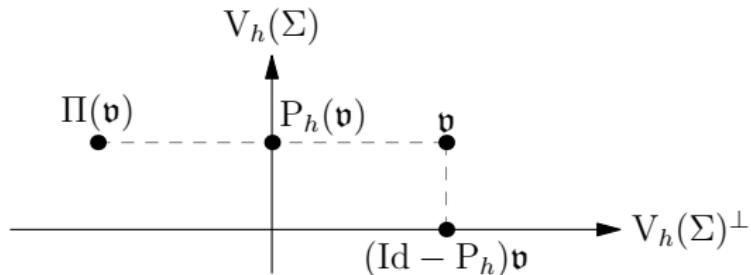
## Matching at interfaces via orthogonal symmetry

The  $t_h$ -orthogonal projection onto the single-trace space  $P_h : \mathbb{V}_h(\Sigma) \rightarrow V_h(\Sigma)$  can be applied by solving a (**DDM friendly!**) SPD problem

$$\begin{aligned} \mathfrak{p} = P_h(\mathfrak{v}) &\iff \mathfrak{p} \in V_h(\Sigma) \quad \text{and} \\ t_h(\mathfrak{p}, \mathfrak{w}) &= t_h(\mathfrak{v}, \mathfrak{w}) \quad \forall \mathfrak{w} \in V_h(\Sigma) \end{aligned}$$

**Lemma :** The  $t_h$ -orthogonal symmetry  $\Pi := P_h - (\text{Id} - P_h) = 2P_h - \text{Id}$  satisfies  $\|\Pi(\mathfrak{q})\|_{t_h} = \|\mathfrak{q}\|_{t_h} \forall \mathfrak{q} \in \mathbb{V}_h(\Sigma)$  and, for any  $\mathfrak{v}, \mathfrak{q} \in \mathbb{V}_h(\Sigma)$ ,

$$(\mathfrak{v}, \mathfrak{q}) \in V_h(\Sigma) \times V_h(\Sigma)^\perp \iff \mathfrak{q} + \imath \mathfrak{v} = \Pi(-\mathfrak{q} + \imath \mathfrak{v}).$$



## Reformulations of the scattering problem

Find  $u_h \in V_h(\Omega)$  and

$$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h(\Omega)$$



$$a(u, v) := \sum_{j=1}^J \int_{\Omega_j} \nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v} d\mathbf{x}$$

$$- \int_{\partial \Omega_j} \imath \kappa u \bar{v} d\sigma$$

$$\ell(v) := \sum_{j=1}^J \int_{\Omega_j} f \bar{v} d\mathbf{x} \quad u, v \in \mathbb{V}_h(\Omega)$$

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**Non trivial theorem :**

- relaxing constraints with Lagrange multipliers
- Use  $\Pi$  to enforce continuity

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**Skeleton formulation**

$$\mathfrak{p}_h \in \mathbb{V}_h(\Sigma) \text{ and}$$

$$(\text{Id} + \Pi S)\mathfrak{p}_h = g_h$$

Unknowns  $u_h$  are eliminated in all subdomains in parallel by local "ingoing-to-outgoing" solves, applying a (block diagonal) scattering operator.

**Proposition :** Define  $S(\mathfrak{p}) := \mathfrak{p} + 2\imath w|_\Sigma$  where  $w \in \mathbb{V}_h(\Omega)$  satisfies  $a(w, v) - \imath t_h(w|_\Sigma, v|_\Sigma) = t_h(\mathfrak{p}, v|_\Sigma) \forall v \in \mathbb{V}_h(\Omega)$ . Then  $\|S(\mathfrak{p})\|_{t_h} \leq \|\mathfrak{p}\|_{t_h}$  for all  $\mathfrak{p} \in \mathbb{V}_h(\Sigma)$ .

## Convergence estimate

**Theorem :**

**1) Boundedness :**  $\|\text{Id} + \Pi S\|_{t_h} \leq 2$

**2) Coercivity :**  $\Re e\{t_h(\mathbf{v}, (\text{Id} + \Pi S)\mathbf{v})\} \geq \gamma_h^2 \|\mathbf{v}\|_{t_h}^2 \quad \forall \mathbf{v} \in \mathbb{V}_h(\Sigma)$

**Coercivity constant**

$$\gamma_h := \frac{\alpha}{\lambda_h^+ + 2 \|a\|/\lambda_h^-}$$

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$$\alpha := \inf_{u_h, v_h \in \mathbb{V}_h(\Omega) \setminus \{0\}} \frac{|a(u_h, v_h)|}{\|u_h\|_{H_\kappa^1(\Omega)} \|v_h\|_{H_\kappa^1(\Omega)}}$$

Coercivity constant

$$\gamma_h := \frac{\alpha}{\lambda_h^+ + 2 \frac{\|a\|/\lambda_h^-}{\|a\|/\lambda_h^-}}$$

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**Theorem :**

1) Boundedness :  $\|\text{Id} + \Pi S\|_{t_h} \leq 2$

2) Coercivity :  $\Re e\{t_h(v, (\text{Id} + \Pi S)v)\} \geq \gamma_h^2 \|v\|_{t_h}^2 \quad \forall v \in \mathbb{V}_h(\Sigma)$

$$\alpha := \inf_{u_h, v_h \in \mathbb{V}_h(\Omega) \setminus \{0\}} \frac{|a(u_h, v_h)|}{\|u_h\|_{H_\kappa^1(\Omega)} \|v_h\|_{H_\kappa^1(\Omega)}}$$

Coercivity constant

$$\gamma_h := \frac{\lambda_h^+ + 2}{\|a\|/\lambda_h^-}$$

$$\|a\| := \sup_{u, v \in H^1(\Omega) \setminus \{0\}} \frac{|a(u, v)|}{\|u\|_{H_\kappa^1(\Omega)} \|v\|_{H_\kappa^1(\Omega)}}$$

$$\lambda_h^+ := \sup_{v_h \in \mathbb{V}_h(\Gamma_j) \setminus \{0\}} \frac{t_{\Gamma_j}(v_h, v_h)}{\|v_h\|_{H_{\kappa, h}^{1/2}(\Gamma_j)}^2}$$

$$\lambda_h^- := \inf_{v_h \in \mathbb{V}_h(\Gamma_j) \setminus \{0\}} \frac{t_{\Gamma_j}(v_h, v_h)}{\|v_h\|_{H_{\kappa, h}^{1/2}(\Gamma_j)}^2}$$

## Convergence estimate

**Theorem :**

1) Boundedness :  $\|\text{Id} + \Pi S\|_{t_h} \leq 2$

2) Coercivity :  $\Re e\{t_h(\mathbf{v}, (\text{Id} + \Pi S)\mathbf{v})\} \geq \gamma_h^2 \|\mathbf{v}\|_{t_h}^2 \quad \forall \mathbf{v} \in \mathbb{V}_h(\Sigma)$

The exact solution  $\mathbf{p}^{(\infty)} \in \mathbb{V}_h(\Sigma)$  to the skeleton formulation can be computed with e.g. a Richardson iteration : given  $r \in (0, 1)$ , compute

$$\mathbf{p}^{(n+1)} = (1 - r)\mathbf{p}^{(n)} - r\Pi S\mathbf{p}^{(n)} + rg_h.$$

**Proposition :** convergence of Richardson's solver

$$\frac{\|\mathbf{p}^{(n)} - \mathbf{p}^{(\infty)}\|_{t_h}}{\|\mathbf{p}^{(0)} - \mathbf{p}^{(\infty)}\|_{t_h}} \leq (1 - 2r(1 - r)\gamma_h^2)^{n/2}.$$

**Important consequence :** If the  $t_{\Gamma_j}(\cdot, \cdot)$ 's yield norms that are  $h$ -uniformly equivalent to  $\|\cdot\|_{H^{1/2}(\Gamma_j)}$ , then we have  **$h$ -uniform geometric convergence**.

## Only 3 hypothesis required

**H1)**  $\Im m\{a(v, v)\} \leq 0 \quad \forall v \in V_h(\Omega)$  i.e. the medium can only absorb or propagate. The case of "purely" propagative media  $\Im m\{a(v, v)\} = 0 \quad \forall v \in V_h(\Omega)$  is covered in particular.

**H2)** Unique solvability of the discrete pb :

$$\inf_{u_h \in V_h(\Omega)} \sup_{v_h \in V_h(\Omega)} \frac{|a(u_h, v_h)|}{\|u_h\|_{H_\kappa^1(\Omega)} \|v_h\|_{H_\kappa^1(\Omega)}} > 0.$$

**H3)** The impedance  $t_h(\cdot, \cdot)$  must be chosen coercive.

No assumption on :

- the shape constant of the mesh
- the frequency regime

# Outline

I Review of the Optimized Schwarz Method

II New manner to enforce transmission conditions

**III Numerical results**

## Numerical experiments : Helmholtz in 2D

Constant wave number  $\kappa > 0$  in a disc  $\Omega = D(0, 1)$  and impedance boundary condition  $(\partial_n - i\kappa)u^{ex} = g$  with  $g(\mathbf{x}) = (\partial_n - i\kappa) \exp(-i\kappa \mathbf{d} \cdot \mathbf{x})$ , discretization with  $V_h(\Omega) = \mathbb{P}_1$ -Lagrange.

$$u_h^{ex} \in V_h(\Omega) \quad \text{and} \quad a(u_h^{ex}, v_h) = \ell(v_h) \quad \forall v_h \in V_h(\Omega)$$

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v} d\mathbf{x} - i\kappa \int_{\partial\Omega} u \bar{v} d\sigma \\ \ell(v) &= \int_{\partial\Omega} \bar{v} g d\sigma. \end{aligned}$$

With  $u_0^{(0)} \equiv 0$ , we denote  $u_h^{(n)}$  the iterates of the linear solver. The measured error is given by

$$(\text{relative error})^2 = \frac{\sum_{j=1}^J \|u_h^{(n)} - u_h^{ex}\|_{H^1(\Omega_j)}^2}{\sum_{j=1}^J \|u_h^{(0)} - u_h^{ex}\|_{H^1(\Omega_j)}^2}.$$

### Remarks :

- global linear solver is GMRes, relative tolerance =  $10^{-8}$
- sequential computations on a 6 core workstation
- FEM & DDM code NIDDL (in Julia) + BemTool (in C++) for integral operators
- exchange operator  $\Pi$  computed with PCG.

## Choices of impedance

Recall that  $t_h(\mathbf{p}, \mathbf{q}) = t_{\partial\Omega_1}(p_1, q_1) + \cdots + t_{\partial\Omega_J}(p_J, q_J)$ . We tested several choices of local impedances.

### Choice 1 : $\mathbf{M} = \text{surface mass matrix}$

$$t_{\partial\Omega_j}(p_h, q_h) = \int_{\partial\Omega_j} p_h(\mathbf{x}) \bar{q}_h(\mathbf{x}) d\sigma(\mathbf{x})$$

This is the impedance originally considered by Després.

### Choice 2 : $\mathbf{K} = \text{surface H1-scalar product}$

$$t_{\partial\Omega_j}(p_h, q_h) = \int_{\partial\Omega_j} \kappa^{-1} \nabla p_h(\mathbf{x}) \cdot \nabla \bar{q}_h(\mathbf{x}) / 2 + \kappa p_h(\mathbf{x}) \bar{q}_h(\mathbf{x}) d\sigma(\mathbf{x})$$

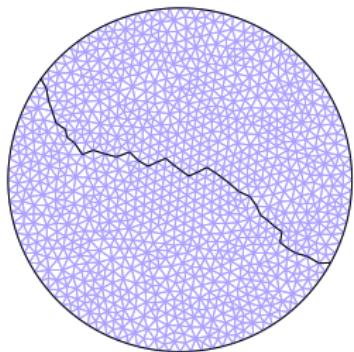
### Choice 3 : $\mathbf{W} = \text{positive hypersingular integral operator}$

$$\begin{aligned} t_{\partial\Omega_j}(p_h, q_h) = & \int_{\partial\Omega_j \times \partial\Omega_j} \exp(-\kappa|\mathbf{x} - \mathbf{y}|) / (4\pi|\mathbf{x} - \mathbf{y}|) [ \\ & \kappa^{-1} \mathbf{n}(\mathbf{x}) \times \nabla_{\partial\Omega_j} p_h(\mathbf{x}) \cdot \mathbf{n}(\mathbf{y}) \times \nabla_{\partial\Omega_j} q_h(\mathbf{y}) \\ & + \kappa \mathbf{n}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{y}) p_h(\mathbf{x}) q_h(\mathbf{y}) ] d\sigma(\mathbf{x}, \mathbf{y}) \end{aligned}$$

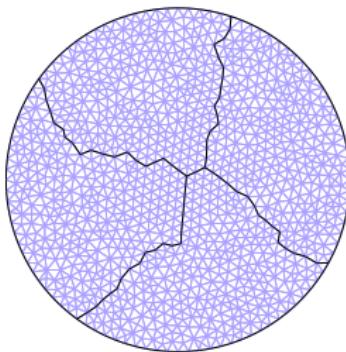
### Choice 4 : $\Lambda = \text{schur complement}$ ( $\simeq$ discrete DtN) associated with the interior numerical solution to the **positive** problem $-\Delta v + \kappa^2 v = 0$ in $\Omega_j$ .

## Mesh partitionning

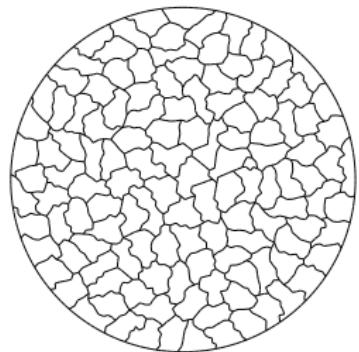
Meshes were generated *a priori* on the whole computational domain with GMSH. Partitionning is obtained *a posteriori* with Metis.



2 subdomains



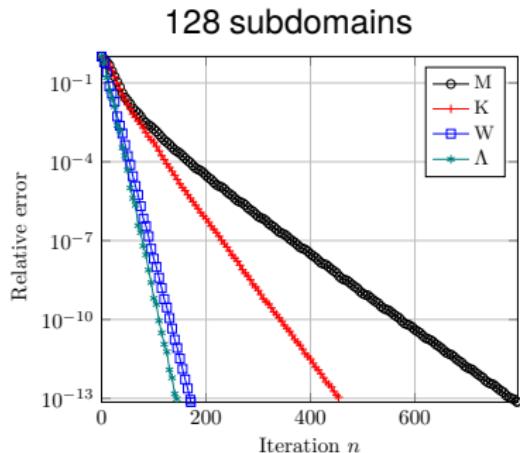
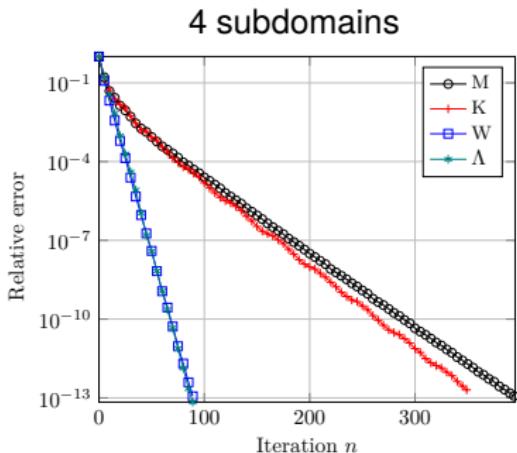
4 subdomains



128 subdomains

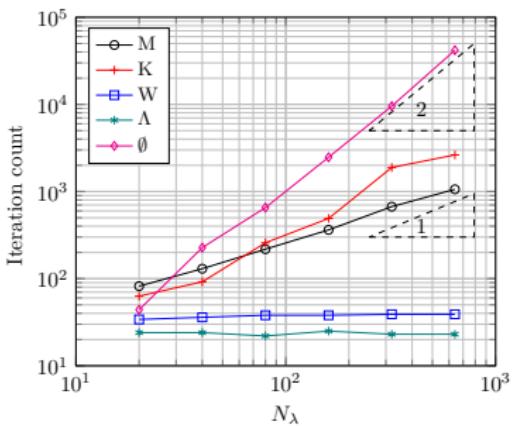
# Convergence history

$\kappa = 5, \quad \lambda = 2\pi/\kappa \simeq 1.25$   
 $N_\lambda = \lambda/h = 40$  points/wavelength.



## Iteration count vs. $N_\lambda$ = points/wavelength

$\kappa = 1$ ,  $\lambda = 2\pi/\kappa \simeq 6.28$ ,  $N_\lambda = \lambda/h$ , 4 subdomains.  
 Relative tolerance of GMRes =  $10^{-8}$ ,  $\emptyset$  = no DDM.

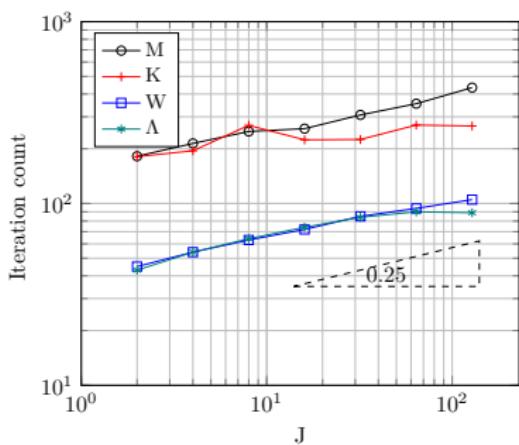


$N_\lambda$	$\emptyset$	M	K	W	$\Lambda$
20	44	82	63	34	24
40	227	130	92	36	24
80	654	218	258	38	22
160	2474	363	491	38	25
320	9559	671	1888	39	23
640	41888	1060	2633	39	23

## Iteration count vs. J = number of subdomains

$$\kappa = 5, \quad \lambda = 2\pi/\kappa \simeq 1.26, \quad N_\lambda = \lambda/h = 40.$$

Relative tolerance of GMRes =  $10^{-8}$ .

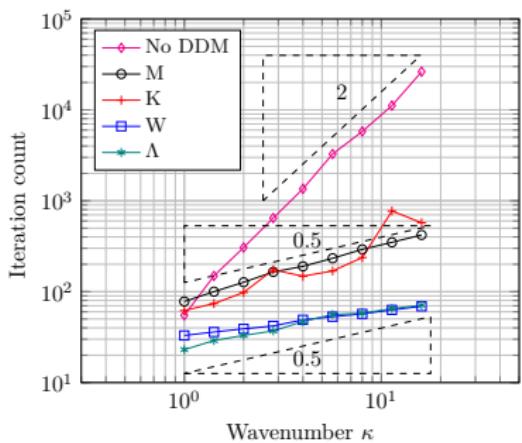


J	M	K	W	Λ
2	182	181	45	43
4	214	195	54	54
8	249	269	63	64
16	258	224	72	74
32	307	225	85	84
64	354	270	94	90
128	434	267	105	89

## Iteration count vs. $\kappa$ = wavenumber

$$h^2 \kappa^3 = (2\pi/20)^2, \quad 4 \text{ subdomains.}$$

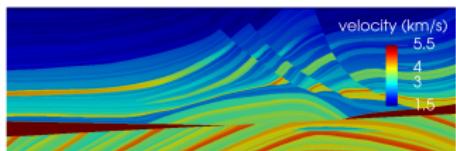
Relative tolerance of GMRes =  $10^{-8}$ ,  $\emptyset$  = no DDM.



$\kappa$	$\emptyset$	M	K	W	$\Lambda$
1	55	78	62	33	23
2	306	127	98	39	33
4	1354	190	148	49	47
8	5791	292	237	57	58
16	26287	421	573	69	71

# Helmholtz 2D Marmousi model

FreeFem + BemTool + HTool

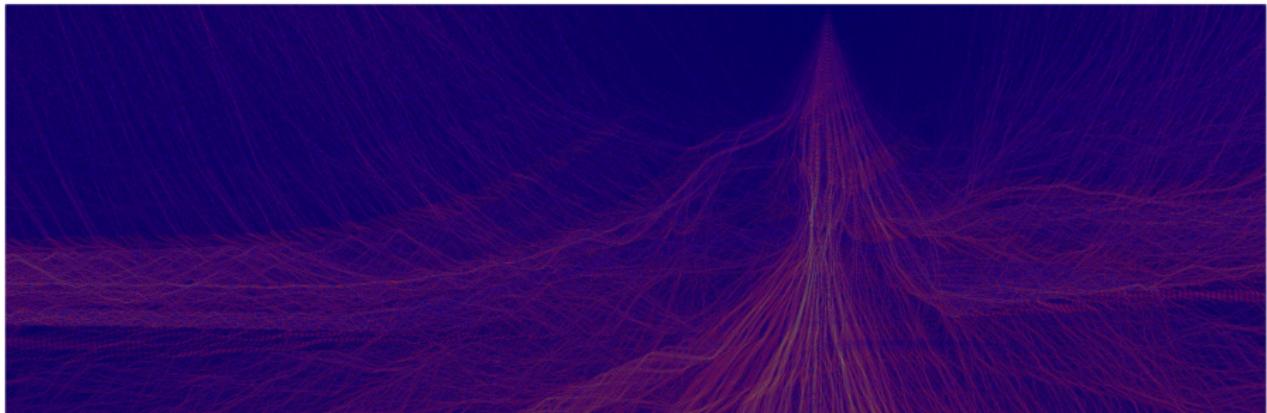


- Computation domain :  $9.2 \times 3$  km
- Impedance =  $W$  = hypersingular op.
- $V_h(\Omega) = \mathbb{P}_2$ -Lagrange
- $N_\lambda = \lambda/h = 8$  points per wavelength
- mesh adapted to the local wavelength
- unstructured partitioning (metis)

$\kappa$	#dofs	J			
		Iteration count			
314	3M	75	134	246	482
628	13M		137	242	464
1256	53M			264	480
4512	213M				510

# Helmholtz 2D Marmousi model

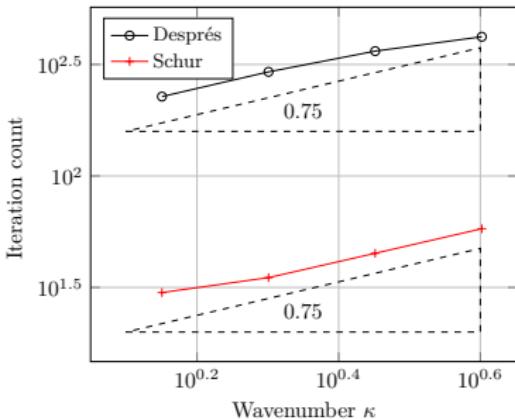
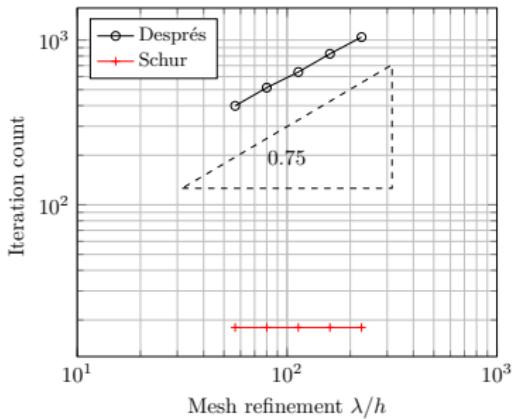
FreeFem + BemTool + HTool



## Maxwell harmonic (3D)

- $\Omega = \text{unit ball}$
- partition (metis) :  $J = 32$  (left) and  $J = 16$  (right)
- discretization : RT0 with  $\kappa^3 h^2 = (2\pi)^2 / 400$
- solved with GMRes

$$\begin{aligned}\operatorname{curl}^2 \mathbf{E} - \kappa^2 \mathbf{E} &= 0 \quad \text{in } \Omega \\ i\kappa \mathbf{n} \times \mathbf{E} \times \mathbf{n} - \operatorname{curl}(\mathbf{E}) \times \mathbf{n} &= \text{plane wave on } \partial\Omega\end{aligned}$$



## **Conclusion**

We proposed a new way of imposing transmission conditions involving another choice of **exchange operator**. This yields ***h*-uniform convergence** of iterative solvers, and accomodates **cross-points**.

In addition this approach appears as a **natural generalization** of the classical OSM à la Després, and allows to propose a **full theoretical framework**, which was not available so far.

### **Also available :**

- other boundary conditions (Dirichlet, Neumann),
- analysis of non-inf-sup-stable impedances.

### **Perspectives**

- fine properties of the exchange operator
- large scale optimized parallel implementation (with **CEA CESTA**)
- multi-level strategy
- non-conforming DDM

# Thank you for your attention

## Questions ?



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X.Claeys and E.Parolin, "Robust treatment of cross points in Optimized Schwarz Methods", accepted in *Numer. Math.*, preprint Arxiv 2003.06657



X.Claeys, "Non-self adjoint impedance in Generalized Optimized Schwarz Methods", submitted, preprint Arxiv 2108.03652



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