A Well-conditioned Weak Coupling of Boundary Element and High-order Finite Element Methods for Time-harmonic Electromagnetic Scattering

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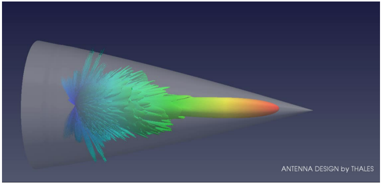
[3]Thales Defense Mission Systems

March 24, 2022

Outline



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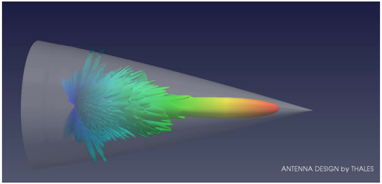

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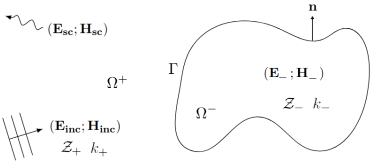

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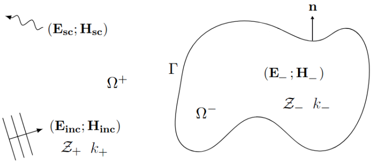
Incident plane wave

$\mathbf{E_{inc}} = (e^{ik_+ . z}, 0, 0)^T$



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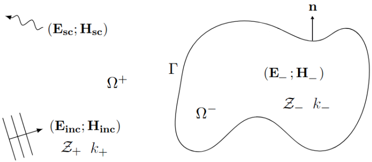

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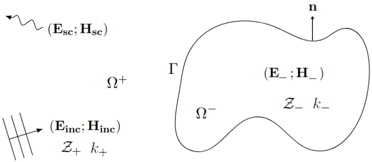
Maxwell harmonic system in $\Omega_\pm$

$$\begin{aligned} \mathbf{curl}\ \mathbf{E}_\pm - \iota k_\pm \mathcal{Z}_\pm \mathbf{H}_\pm &= \mathbf{0} \\ \mathbf{curl}\ \mathbf{H}_\pm + \iota k_\pm \mathcal{Z}_\pm^{-1} \mathbf{E}_\pm &= \mathbf{0} \end{aligned}$$



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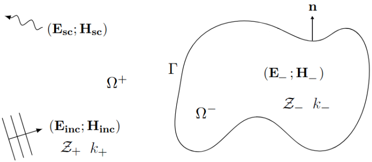
Silver-Müller radiation condition

$$\mathcal{Z}_+ \mathbf{H_{sc}} \times \frac{\mathbf{x}}{||\mathbf{x}||} - \mathbf{E_{sc}} = \underset{r \to +\infty}{\mathcal{O}}(r^{-2})$$



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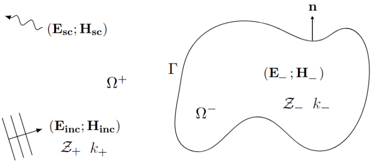
Transmission conditions on Γ

$$\begin{aligned} \mathbf{E}_{-|\Gamma} \times \mathbf{n} &= \mathbf{E}_{\mathbf{sc}|\Gamma} \times \mathbf{n} + \mathbf{E}_{\mathbf{inc}|\Gamma} \times \mathbf{n} \\ \mathbf{H}_{-|\Gamma} \times \mathbf{n} &= \mathbf{H}_{\mathbf{sc}|\Gamma} \times \mathbf{n} + \mathbf{H}_{\mathbf{inc}|\Gamma} \times \mathbf{n} \end{aligned}$$

with $\mathbf{n}$ the outward pointing normal on Γ



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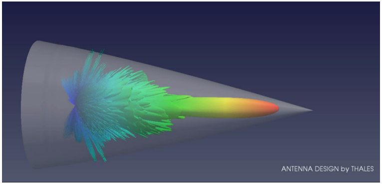
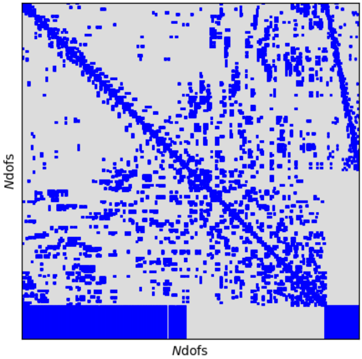
$\rightarrow$ Two standard approaches to overcome this difficulty:

- Truncation methods (PML or ABC): approximate the scattering problems at the continuous level.
- **Surface integral equations for the exterior domain**: exact at the continuous level and adapted to homogeneous domain.

Outline



Strong Coupling, a Standard Resolution Method



Matrix resulting from the coupling of variational formulation for Ω_- and an integral equation for Ω_+ in one unique formulation.

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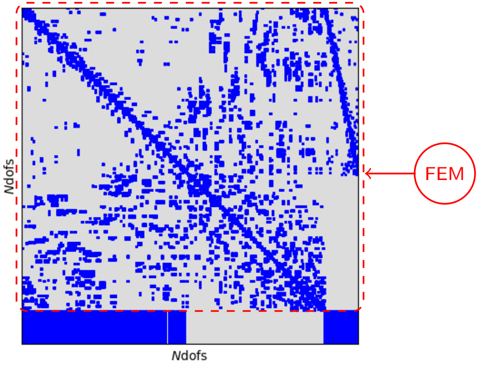

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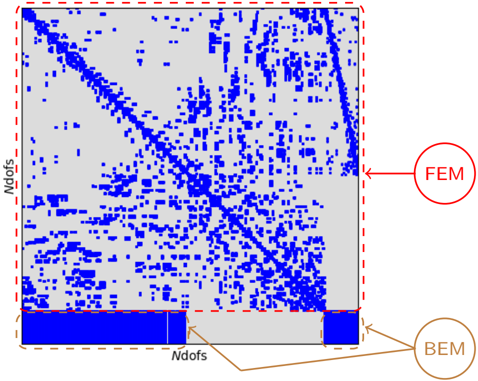

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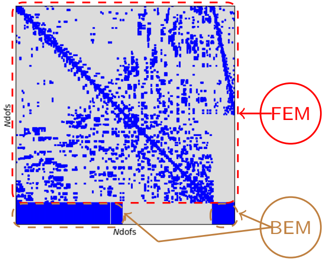
Difficulty for classical iterative methods

- Strong FEM-BEM leads to a matrix with sparse and dense blocks.



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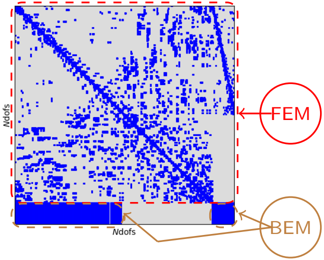

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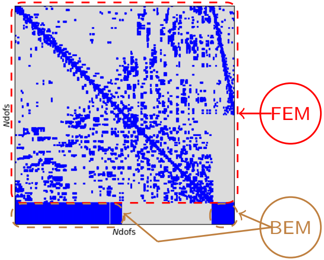

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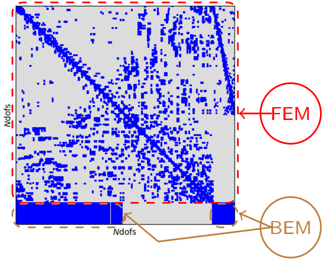

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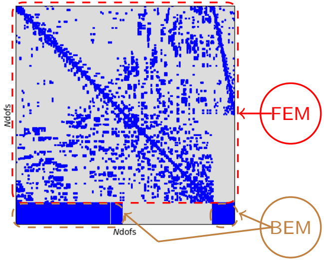
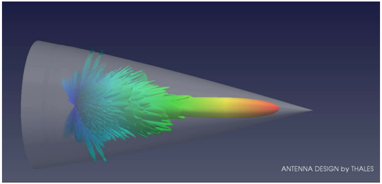


A simple solution to all these problems

Domain decomposition between the interior domain and the exterior domain.

Outline



What is the Weak Coupling?

It is a Schwarz-like method: reformulate the transmission conditions on Γ

$$
\begin{aligned}
(\mathbf{H}_{-|\Gamma} \times \mathbf{n}) + \mathbf{T}_{-}(\mathbf{E}_{-|\Gamma} \times \mathbf{n}) &= (\mathbf{H}_{+|\Gamma} \times \mathbf{n}) + \mathbf{T}_{-}(\mathbf{E}_{+|\Gamma} \times \mathbf{n}) \\
(\mathbf{H}_{+|\Gamma} \times \mathbf{n}) - \mathbf{T}_{+}(\mathbf{E}_{+|\Gamma} \times \mathbf{n}) &= (\mathbf{H}_{-|\Gamma} \times \mathbf{n}) - \mathbf{T}_{+}(\mathbf{E}_{-|\Gamma} \times \mathbf{n})
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Unknowns of the weak coupling

$$\mathbf{g}_{\pm} = (\mathbf{H}_{\pm|\Gamma} \times \mathbf{n}) \mp \mathbf{T}_{\pm}(\mathbf{E}_{\pm|\Gamma} \times \mathbf{n})$$

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Affine resolution operators (the pre-existing solvers)

$$\mathbf{R}_{-}\mathbf{g} = \mathbf{E}_{-|\Gamma} \times \mathbf{n} \ (\texttt{FEM solver}) \text{ and } \mathbf{R}_{+}\mathbf{g} = \mathbf{E}_{+|\Gamma} \times \mathbf{n} \ (\texttt{BEM solver})$$

where $(\mathbf{E}_{\pm}; \mathbf{H}_{\pm})$ are the solutions of the following boundary-value problems:

$$\begin{aligned}\textbf{curl } \mathbf{E}_{\pm} - \iota k_{\pm}\mathcal{Z}_{\pm}\mathbf{H}_{\pm} &= \mathbf{0}, \text{ in } \Omega_{\pm} \\ \textbf{curl } \mathbf{H}_{\pm} + \iota k_{\pm}\mathcal{Z}_{\pm}^{-1}\mathbf{E}_{\pm} &= \mathbf{0}, \text{ in } \Omega_{\pm} \\ (\mathbf{H}_{\pm|\Gamma} \times \mathbf{n}) \mp \mathbf{T}_{\pm}(\mathbf{E}_{\pm|\Gamma} \times \mathbf{n}) &= \mathbf{g}, \text{ on } \Gamma \\ \mathcal{Z}_{+}\mathbf{H}_{\mathbf{sc}} \times \frac{\mathbf{x}}{\|\mathbf{x}\|} - \mathbf{E}_{\mathbf{sc}} &= \underset{r \to +\infty}{\mathcal{O}}(r^{-2}).\end{aligned}$$

The Weak FEM-BEM Coupling

$\rightarrow$ Rewriting of the reformulated transmission conditions

The weak FEM-BEM coupling

$$(\mathbf{Id} - \mathcal{S}_\pi)\begin{pmatrix}\mathbf{g}_- \\ \mathbf{g}_+\end{pmatrix} = \begin{pmatrix}\mathbf{0} \\ \mathbf{0}\end{pmatrix}$$

$$\text{with } \boldsymbol{\mathcal{S}_\pi} = \begin{pmatrix} 0 & \mathbf{S}_+ \\ \mathbf{S}_- & 0 \end{pmatrix},\ \mathbf{S}_\pm = \mathbf{Id} \pm (\mathbf{T}_- + \mathbf{T}_+)\mathbf{R}_\pm$$

Reformulation of the Weak Coupling

Reformulation of the weak FEM-BEM coupling

The weak FEM-BEM coupling can be recast into a linear system:

$$(\mathbf{Id} - \mathcal{A}) \begin{pmatrix} \mathbf{g}_- \\ \mathbf{g}_+ \end{pmatrix} = \begin{pmatrix} \mathbf{b}_- \\ \mathbf{b}_+ \end{pmatrix} (\clubsuit),$$

where the information about the incident plane wave is contained in the right-hand side. The quantities $\mathbf{g}_-$ and $\mathbf{g}_+$ can be interpreted as some exchanged information from Ω_+ to Ω_- and from Ω_- to Ω_+, respectively.

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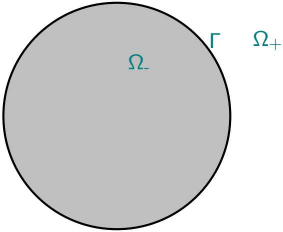
At each iteration of a **Krylov subspace solver** of ($\clubsuit$):



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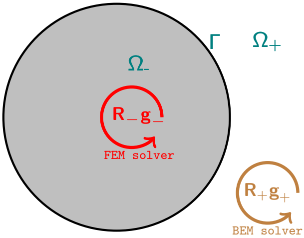

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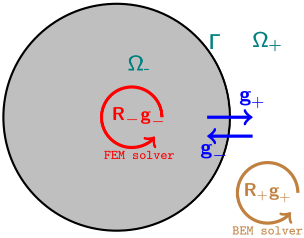
At each iteration of a **Krylov subspace solver** of ($\clubsuit$):

1. Solve the interior problem and the exterior problem.
2. Update the interface unknowns $\mathbf{g}_+$ and $\mathbf{g}_-$ (using $\boldsymbol{\mathcal{A}}$).



Correctly Choosing the Transmission Operators

The weak coupling can **only be solved iteratively**, typically using the GMRES method.
$\rightarrow$The GMRES convergence of the weak coupling is fundamentally related to the choice of the transmission operators.

Optimal transmission operators for the weak coupling

[Caudron, Geuzaine, Antoine, 2018]

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\begin{gathered}
\mathbf{T}_{-} = -\boldsymbol{\Lambda}_{+,k_{+},\mathcal{Z}_{+}} \Rightarrow \widetilde{\mathbf{S}}_{+} = \mathbf{0} \\
\mathbf{T}_{+} = \boldsymbol{\Lambda}_{-,k_{-},\mathcal{Z}_{-}} \Rightarrow \widetilde{\mathbf{S}}_{-} = \mathbf{0} \\
\Rightarrow Sp(\mathbf{Id} - \boldsymbol{\mathcal{A}}) = \{1\}
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$$\mathbf{T}_+ = \mathbf{\Lambda}_{-,k_-,\mathcal{Z}_-} \Rightarrow \widetilde{\mathbf{S}}_- = \mathbf{0}$$
$$\Rightarrow Sp(\mathbf{Id} - \boldsymbol{\mathcal{A}}) = \{1\}$$

Remarks: Unlike the interior problem (complex **nonlocal** interactions), the solution of the exterior domain is **spatially localized**.
$\rightarrow$ Use accurate representation of $\mathbf{T}_- = -\mathbf{\Lambda}_{+,k_+,\mathcal{Z}_+}$ and rough computation of $\mathbf{T}_+$ ($\approx$ poor approximation of $-\mathbf{\Lambda}_{+,k_+,\mathcal{Z}_+}$ or $\mathbf{\Lambda}_{-,k_-,\mathcal{Z}_-}$).

$\Rightarrow \boldsymbol{\mathcal{A}}$ nilpotent (leading fast convergence of the GMRES)

Approximating The **MtE** Operators

What about the practical side?

In practice, evaluating **MtE** operators leads to high computational costs. Several practical localized approximations of the **MtE** operators were proposed in literature.

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Padé-localized square-root approximations [El Bouajaji, Antoine, Geuzaine, 2014]

$$\mathbf{\Lambda}^{sq,N_p,\theta_p}_{\pm,k_\pm,\mathcal{Z}_\pm} = \mp \frac{1}{\mathcal{Z}_\pm}\left(C_0 + \sum_{l=1}^{N_p} A_l \left(\nabla_\Gamma(\tfrac{1}{k_\epsilon^2} div_\Gamma) - \mathbf{curl}_\Gamma(\tfrac{1}{k_\epsilon^2} curl_\Gamma)\right)\left(\mathbf{Id} + B_l\left(\nabla_\Gamma(\tfrac{1}{k_\epsilon^2} div_\Gamma) - \mathbf{rot}_\Gamma(\tfrac{1}{k_\epsilon^2} rot_\Gamma)\right)\right)^{-1}\right)^{-1} \left(\mathbf{Id} - \mathbf{curl}_\Gamma(\tfrac{1}{k_\epsilon^2} curl_\Gamma)\right)(\mathbf{Id} \times \mathbf{n})$$

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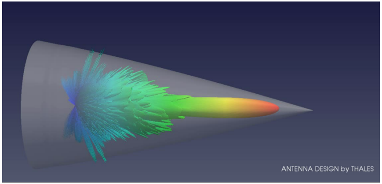


Numerical Results

FEM solver

`GmshFEM`,
a newly developed open-source finite element library based on `Gmsh`

BEM solver

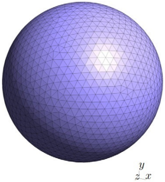
`Bempp-cl`,
an open-source boundary element library

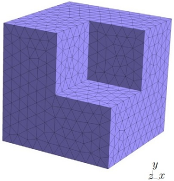
The transmission operators selected

$$(\mathbf{T}_{-}; \mathbf{T}_{+})$$

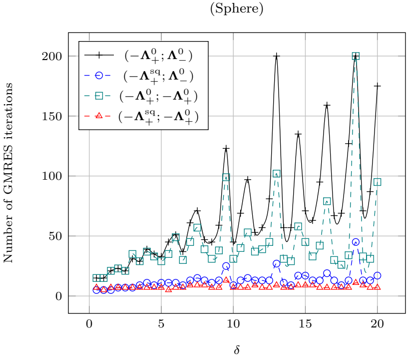
We suppose

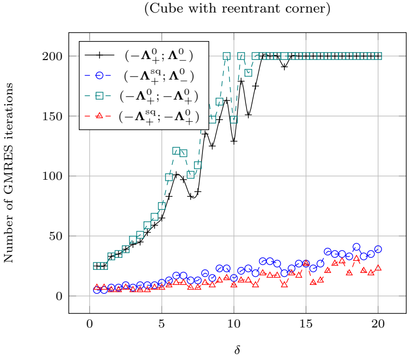
$$k_{-} = \delta k_{+} e^{-\|\mathbf{x}\|^2} \text{ and } \mathcal{Z}_{-} = \frac{\mathcal{Z}_{+}}{\delta} e^{\|\mathbf{x}\|^2} \text{ with } \delta \in \mathbb{R}_{+}^{*} \text{ and } \mathcal{Z}_{+} = \mathcal{Z}_{0}$$





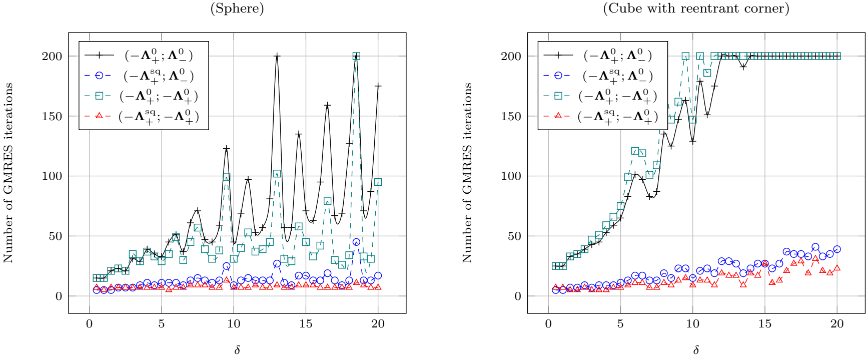
Number Of GMRES Iterations *Vs.* δ





5 points per wavelength and $k_+ = 1$

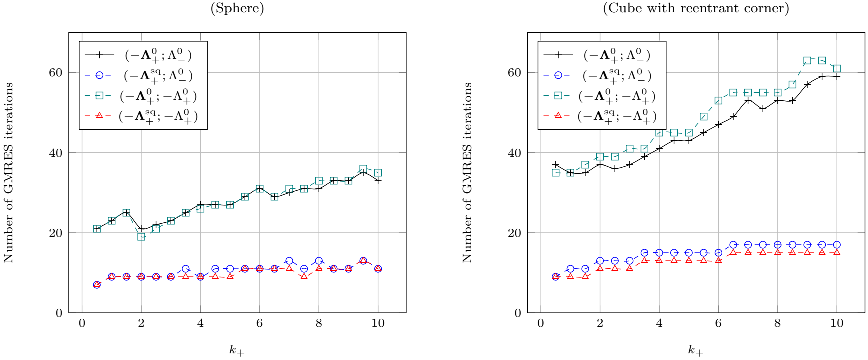
Number Of GMRES Iterations *Vs.* δ



5 points per wavelength and $k_+ = 1$

$\rightarrow$ **Superior iterative performance** when $\mathbf{T}_+ = -\mathbf{\Lambda}^0_{+,k_+,\mathcal{Z}_+}$

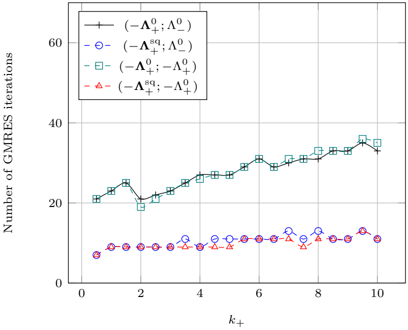
Number Of GMRES Iterations *Vs.* k_+



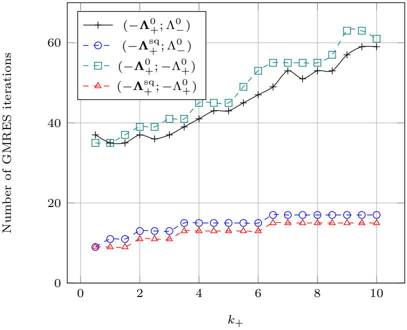
5 points per wavelength, $\delta = 2$

Number Of GMRES Iterations *Vs.* k_+

(Sphere)



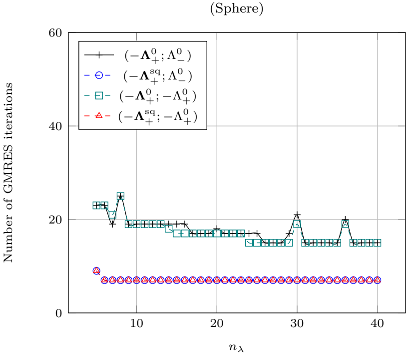
(Cube with reentrant corner)

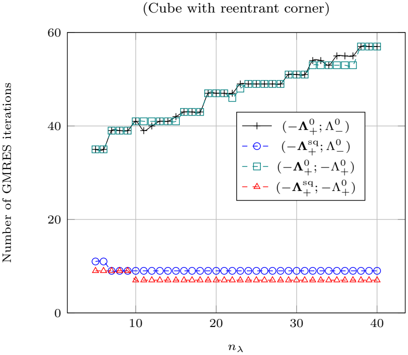


5 points per wavelength, $\delta = 2$

$\rightarrow (-\mathbf{\Lambda}^{sq,N_p,\theta_p}_{+,k_+,\mathcal{Z}_+}; -\mathbf{\Lambda}^0_{+,k_+,\mathcal{Z}_+})$ leads to the **lowest dependency** of the GMRES iterations with respect to k_+

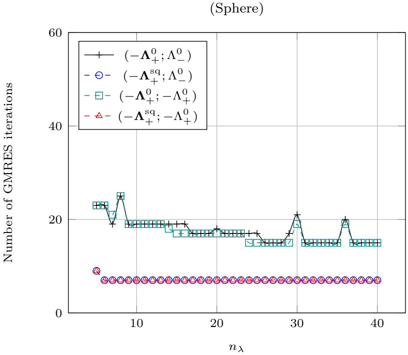
Influence Of Mesh Refinement

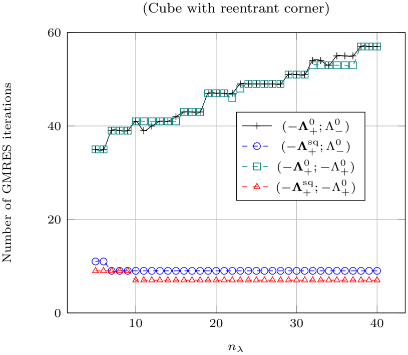




$\delta = 2$ and $k_+ = 1$

Influence Of Mesh Refinement





$\delta = 2$ and $k_+ = 1$

$\rightarrow$ Convergence clearly **deteriorates** with the pairs $(-\mathbf{\Lambda}^0_{+,k_+,\mathcal{Z}_+};\mp\mathbf{\Lambda}^0_{\pm,k_\pm,\mathcal{Z}_\pm})$

Accuracy of the weak FEM-BEM coupling

Relative differences in $\mathbf{L}_t^2(\Gamma)$-norm between the strong and the weak FEM-BEM coupling electric/magnetic currents

$$e_{\mathbf{H}_{-,h_{|\Gamma}} \times \mathbf{n}} = \frac{\left\| \mathbf{H}_{-,h_{|\Gamma}} \times \mathbf{n} - \mathbf{H}_{|\Gamma}^{\mathrm{Ref}} \times \mathbf{n} \right\|_{\mathbf{L}_t^2(\Gamma_h)}}{\left\| \mathbf{H}_{|\Gamma}^{\mathrm{Ref}} \times \mathbf{n} \right\|_{\mathbf{L}_t^2(\Gamma_h)}},$$

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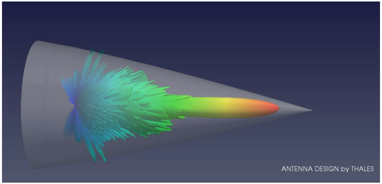
k_+	δ	$e_{\mathbf{E}_{-,h_{\|\Gamma}}\times\mathbf{n}}$	$e_{\mathbf{H}_{-,h_{\|\Gamma}}\times\mathbf{n}}$
1	2	9.9%	0.8%
2	2	7.5%	0.8%
1	3	5.7%	4.9%
1	4	2.9%	1.0%

$e_{\mathbf{E}_{-,h_{|\Gamma}}\times\mathbf{n}}$ and $e_{\mathbf{H}_{-,h_{|\Gamma}}\times\mathbf{n}}$ for the sphere and for different k_+ and δ ($n_\lambda = 10$)

Outline



Applications

FEM solver

`GmshFEM`

BEM solver

`Antenna Design` (`AD`) coupled with the H-Matrix library `Hi-BoX`,
the Thales internally developed code

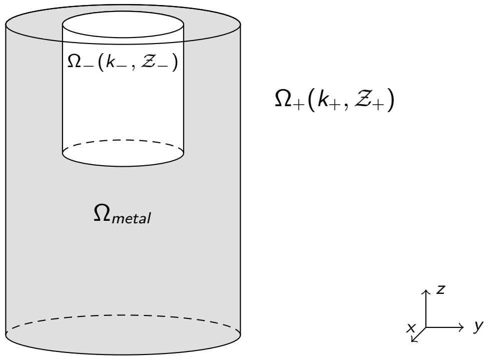
Validation

The test cases were validated using `AD` coupled with `Hi-BoX`

The transmission operators selected

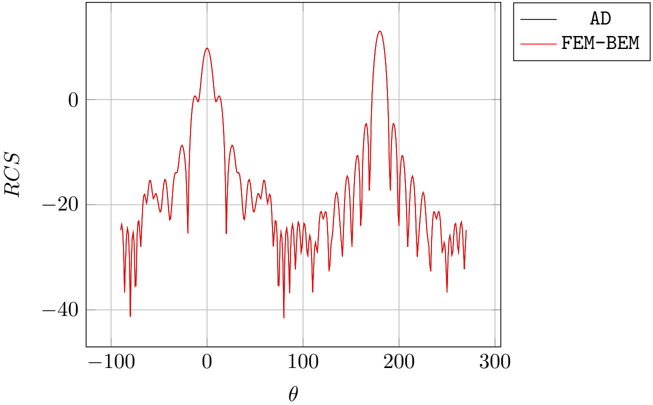
$$(\mathbf{T}_-;\mathbf{T}_+) = (-\mathbf{\Lambda}_+^{\mathrm{sq}}; -\mathbf{\Lambda}_+^0)$$

Partially coated domain



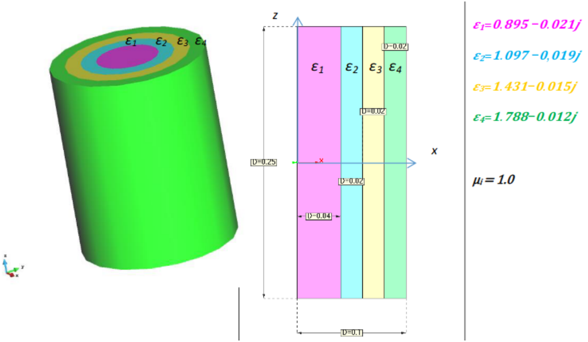
- Incident plane wave linearly polarized along z
- Frequency: $10[GHz]$
- $\epsilon_- = (0.895 - \iota 0.021)\epsilon_0$, $\mu_- = \mu_0$,
- $\epsilon_+ = \epsilon_0$, $\mu_+ = \mu_0$

The bistatic RCS in plane $y = 0$



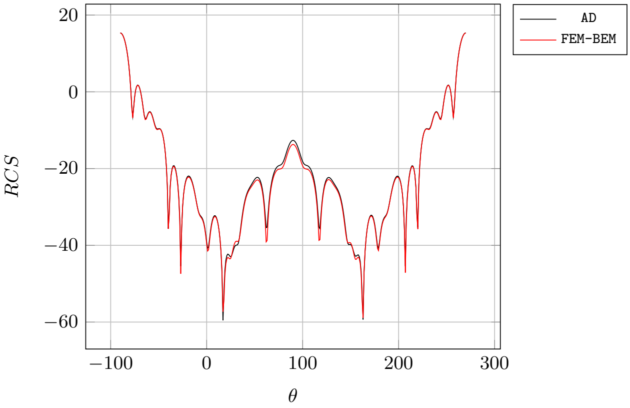
Good convergence of GMRES in 11 iterations

Multilayer Dielectric Cylinder



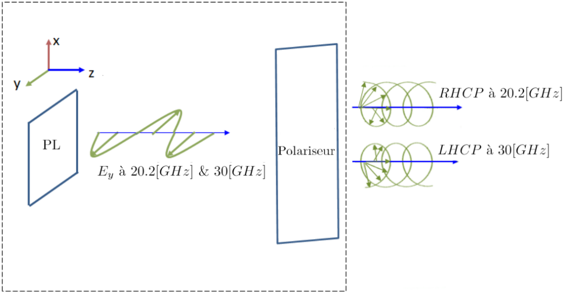
- Incident plane wave linearly polarized along x
- Frequency: $5[GHz]$

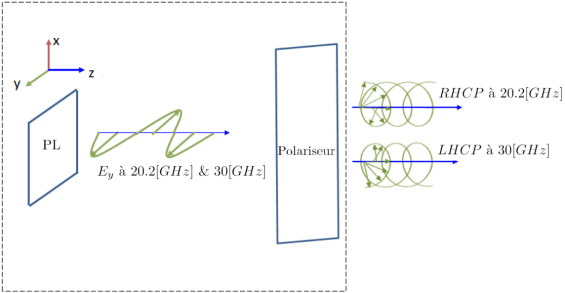
The bistatic RCS in plane $y = 0$



Good convergence of GMRES in 15 iterations

A K/KA Dual Band Polarizer





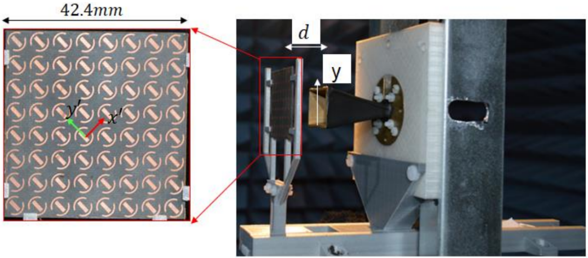
We compute the RHCP gain and the LHCP gain in the *yz* plane defined by

$$RHCP(\theta, 90) = 20\log(|\frac{1}{\sqrt{2}}(\mathsf{E}^{\infty}_{\mathsf{sc},\theta}(\theta, 90) + \iota \mathsf{E}^{\infty}_{\mathsf{sc},\phi}(\theta, 90)|),$$

$$LHCP(\theta, 90) = 20\log(|\frac{1}{\sqrt{2}}(\mathsf{E}^{\infty}_{\mathsf{sc},\theta}(\theta, 90) - \iota \mathsf{E}^{\infty}_{\mathsf{sc},\phi}(\theta, 90)|),$$

where we denote the components of the far-field by $\mathsf{E}^{\infty}_{\mathsf{sc},\theta}$ and $\mathsf{E}^{\infty}_{\mathsf{sc},\phi}$

A K/KA Dual Band Polarizer



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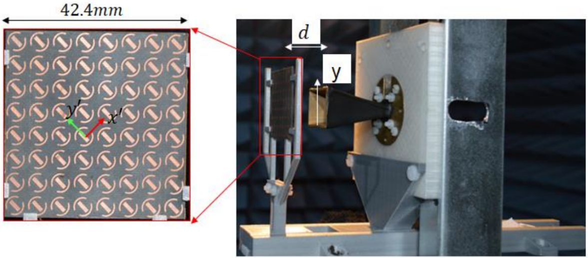

- The polarizer is tested with two linearly polarized rectangular horns, one for each band (20.2[GHz] and 30[GHz])
- This problem can be challenging due to the rather small thickness and intricacies of the polarizer
- For more information: P. Naseri et al, *Dual-Band Dual-Linear-to-Circular Polarization Converter in Transmission Mode Application to K/Ka -Band Satellite Communications*, in IEEE, 2018.

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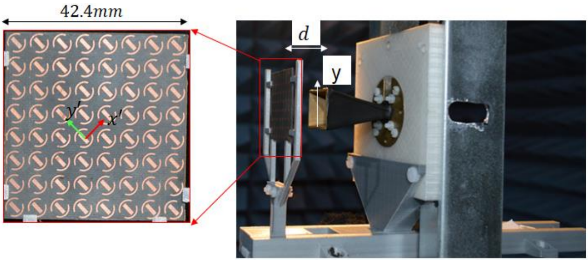

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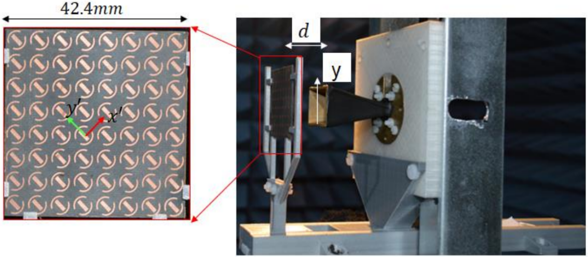

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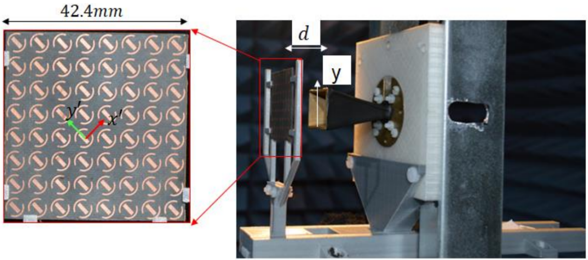

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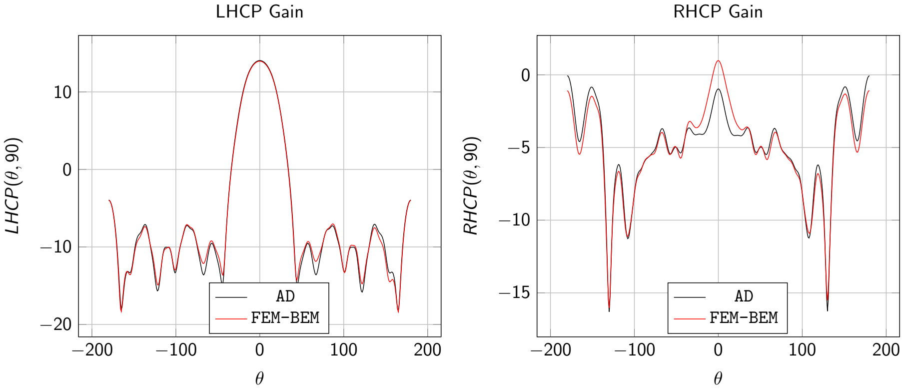
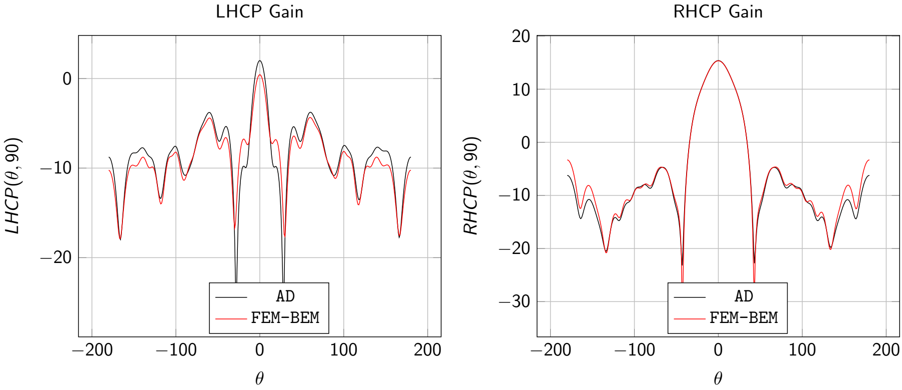
$30[GHz]$

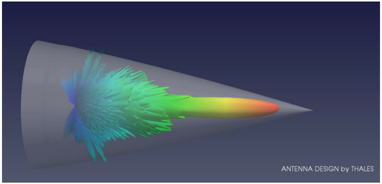


Good convergence of GMRES in 15 iterations



Good convergence of GMRES in 17 iterations

Outline



Conclusion

Features of the new weak FEM-BEM coupling

- The convergence rate is **slightly dependent** on the geometry configuration, the frequency k, the mesh refinement and the contrast between the two subdomains.
- Allows to reuse optimized pre-existing solvers.
- For a relevant accuracy on the far-field, the number of iterations is very small and stable (typically about 15).

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Perspectives

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- DDM for FEM part.

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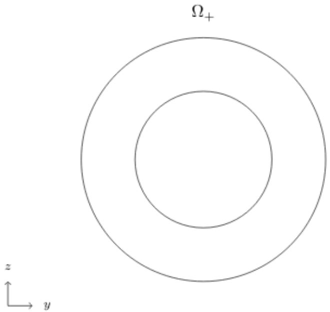
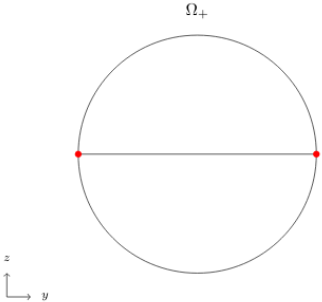


(a) Concentric partition



(b) Partition with corners

Perspectives

	Concentric partition	Partition with junctions
GMRES iterations	9	30
$e_{\mathbf{E}_{-,h_{\vert\Gamma}} \times \mathbf{n}}$	1.8%	3.8%
$e_{\mathbf{H}_{-,h_{\vert\Gamma}} \times \mathbf{n}}$	0.5%	2.6%

I. Badia, C. Caudron, X. Antoine and C. Geuzaine. *A well-conditioned weak coupling of boundary element and high-order finite element methods for time-harmonic electromagnetic scattering by inhomogeneous objects*, to be published in SIAM Journal on Scientific Computing, 2022

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Thank you for your attention !

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